

# Gap Probabilities for Double Intervals in Hermitian Random Matrix Ensembles as $\tau$ -Functions – Spectrum Singularity case

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The probability for the exclusion of eigenvalues from an interval  $(-x, x)$  symmetrical about the origin for a scaled ensemble of Hermitian random matrices, where the Fredholm kernel is a type of Bessel kernel with parameter  $a$  (a generalisation of the sine kernel in the bulk scaling case), is considered. It is shown that this probability is the square of a  $\tau$ -function, in the sense of Okamoto, for the Painlevé system  $P_{III}$ . This then leads to a factorisation of the probability as the product of two  $\tau$ -functions for the Painlevé system  $P_{III'}$ . A previous study has given a formula of this type but involving  $P_{III'}$  systems with different parameters consequently implying an identity between products of  $\tau$ -functions or equivalently sums of Hamiltonians.

The probability  $E_\beta(0; J; g(x); N)$  that a subset of the real line  $J$  is free of eigenvalues for an ensemble of  $N \times N$  random matrices with eigenvalue probability density function proportional to

$$(1) \quad \prod_{l=1}^N g(x_l) \prod_{1 \leq j < k \leq N} |x_j - x_k|^\beta,$$

( $\beta = 1, 2$  or  $4$  according to the ensemble exhibiting orthogonal, unitary or symplectic symmetry respectively) is a fundamental statistic in the study of these ensembles. Most effort has focused on the case where  $J$  is a single interval, one endpoint fixed at the edge of the support of the measure defining the ensemble whilst the other is free, and taken to be the independent variable in the system of equations determining the gap probability. There is, however, another interesting case where the set  $J$  consists of two disconnected intervals, but related to each other so that there is still only one free variable. For instance there is the result for unitary ensembles of Hermitian matrices that the gap probability for an interval symmetrical about the origin  $J$  and an even weight function  $g_2(x)$  (where the integrable examples include Gaussian, symmetric Jacobi and Cauchy weights) factorises [4]

$$(2) \quad E_2(0; J; g_2(x); N) = E_2(0; J^+; y^{-1/2} g_2(y^{1/2}); \lfloor (N+1)/2 \rfloor) E_2(0; J^+; y^{1/2} g_2(y^{1/2}); \lfloor N/2 \rfloor),$$

( $J^+$  is the positive member of a pair of intervals composing  $J$ ). Examples of where this relation has been useful can be found in the above reference and [8]. An example of such a double interval statistic is one arising from an ensemble of random unitary  $N \times N$  matrices with the joint eigenvalue pdf

$$(3) \quad p(z_1, \dots, z_N) = C_{N,a} \prod_{l=1}^N |1 - z_l|^{2a} \prod_{1 \leq j < k \leq N} |z_j - z_k|^2, \quad z_l = e^{i\theta_l}.$$

The weight function has an algebraic singularity at  $z = e^{i\theta} = 1$  corresponding in the log-gas picture of (3) to an impurity charge of  $a$ , so the ensemble is termed the *spectrum singularity* case. The probability that no eigenvalues  $e^{i\theta}$  have phases in the interval  $(-\theta, \theta)$  is given by

$$(4) \quad E_2(0; (-\theta, \theta); |1 - z|^{2a}; N) \\ = \int_{(-\pi, -\theta) \cup (\theta, \pi)} \frac{dz_1}{2\pi i z_1} \cdots \int_{(-\pi, -\theta) \cup (\theta, \pi)} \frac{dz_N}{2\pi i z_N} p(z_1, \dots, z_N).$$

This problem can also be equivalently expressed in terms of a problem with its spectrum on the real line, where the weight function is now a Cauchy one, and the above gap probability is  $E_2(0; (-\infty, -s) \cup (s, \infty); (1 + \lambda^2)^{-N-a}; N)$ .

We wish to focus here on the specific example of a double interval  $J = (-x, x)$  for the bulk scaling limit of the Spectrum Singularity case, as  $N \rightarrow \infty$ . The gap probability is known to be expressed as a Fredholm determinant

$$(5) \quad E_2^{\text{SS}}(0; (-x, x); a) = \det(1 - \mathbb{K}_J),$$

where the integral operator  $\mathbb{K}$  has the kernel

$$(6) \quad K(x, y) = \sqrt{\pi x} \sqrt{\pi y} \frac{J_{a+1/2}(\pi x) J_{a-1/2}(\pi y) - J_{a+1/2}(\pi y) J_{a-1/2}(\pi x)}{2(x - y)},$$

with a parameter  $a \in \mathbb{C}$  with  $\text{Re}(a) > -1/2$  and the density of eigenvalues is  $\rho = 1$ . The Tracy and Widom theory for the Fredholm determinant forms of gap probabilities [18] was employed in [5, 19], and this probability was evaluated as

$$(7) \quad E_2^{\text{SS}}(0; (-x, x); a) = \exp \left( \int_0^{2\pi x} dy \frac{\sigma_1(y)}{y} \right).$$

in terms of  $\sigma_1(r) \equiv -2xR(x, x)$  with  $r = 2x$ , which was shown to satisfy the ordinary differential equation

$$(8) \quad (r\sigma_1'')^2 + 4[-a^2 - \sigma_1 + r\sigma_1'] \left\{ (\sigma_1')^2 - \left[ a - \sqrt{a^2 + \sigma_1 - r\sigma_1'} \right]^2 \right\} = 0,$$

subject to the boundary condition

$$(9) \quad \sigma_1(r) \underset{r \rightarrow 0^+}{\sim} \\ C_a r^{2a+1} \left( 1 - \frac{a}{2(2a+3)(2a+1)} r^2 + \frac{a}{16(2a+5)(2a+3)(2a+1)} r^4 + \dots \right) \\ - C_a^2 \frac{r^{4a+2}}{2a+1} \left( 1 - \frac{a+1}{(2a+3)^2} r^2 + \dots \right) + C_a^3 \frac{r^{6a+3}}{(2a+1)^2} (1 + \dots) + \dots,$$

where

$$(10) \quad C_a = -\frac{2}{4^{2a+1} \Gamma(a + \frac{1}{2}) \Gamma(a + \frac{3}{2})}.$$

When  $a = 0$  this equation reduces to case of the bulk scaling limit first found in [13], and is then a special case of the Jimbo-Miwa-Okamoto  $\sigma$ -function form of  $P_V$ . Equation (8) was solved in terms of Painlevé's fifth transcendent [19] for general  $a$  with the parameters

$$(11) \quad \alpha = \frac{1}{32}(1 - 2a)^2, \quad \beta = -\frac{1}{32}(1 - 2a)^2, \quad \gamma = 0, \quad \delta = -2.$$

However there is a puzzling aspect to this result, which also appears in the cases of double intervals with other weights such as the Gaussian, symmetric Jacobi [20] or Cauchy [19] ones. Expressed as a second-order second-degree ODE (8) has radical terms, namely the square-root term, or alternatively is a quartic second-order ordinary differential equation in polynomial form. Consequently, the ordinary differential equation (8) is not of the Jimbo-Miwa-Okamoto  $\sigma$ -form of  $P_V$  for  $a \neq 0$ , as is the case for the single intervals. On this point we remark there is now a large body of works [6, 7, 9] demonstrating that gap probabilities and their generalised averages for unitary random matrix ensembles with classical weights can be evaluated in terms of a single  $\tau$ -function for one of the Painlevé systems. Consequently the logarithmic derivatives of these averages satisfy a Jimbo-Miwa-Okamoto  $\sigma$ -form for the appropriate system, which are generically only second-order second-degree ordinary differential equations (see the classification of such ODEs with the Painlevé property by Cosgrove and Scoufis [1]). Therefore it was not clear how the spectrum singularity result fitted into this broader scheme. We provide an answer to this question in this work - there is a natural explanation in the Okamoto theory for the Painlevé transcendents  $P_{III}$  and  $P_{III'}$ .

Firstly we recognise that the special case (11) of  $P_V$  is one that degenerates to  $P_{III}$  using the transformations of Gromak [11, 12] - for example this can be achieved with the parameters  $(v_1, v_2) = (a - 1/2, a - 1/2)$  or  $\alpha_{III} = 1 - 2a, \beta_{III} = 1 + 2a, \gamma_{III} = 1, \delta_{III} = -1$ . Equation (8) arises quite naturally in the Painlevé III system, as will be apparent from Okamoto's theory for  $P_{III}$  [15]. The  $P_{III}$  differential equation is

$$(12) \quad q'' = \frac{1}{q} (q')^2 - \frac{1}{t} q' + \frac{1}{t} (\alpha q^2 + \beta) + \gamma q^3 + \frac{\delta}{q},$$

and can be generated from the Hamiltonian system

$$(13) \quad tH_{III} = 2q^2 p^2 - [2\eta_\infty t q^2 + (2v_1 + 1)q - 2\eta_0 t] p + \eta_\infty (v_1 + v_2) t q.$$

One then recovers the standard  $P_{III}$  ODE for  $q(t)$  with the parameter identifications

$$(14) \quad \alpha = -4\eta_\infty v_2, \beta = 4\eta_0 (v_1 + 1), \gamma = 4\eta_\infty^2, \delta = -4\eta_0^2.$$

Here  $\eta_\infty, \eta_0$  are arbitrary parameters which control the scaling of the independent and dependent variables, so they are usually fixed at some nominal value (unity). Now let us define the auxiliary Hamiltonian

$$(15) \quad h = tH_{III} + 1/8(2v_1 + 1)^2,$$

and examine the time evolution of this.

**Theorem 1** (Proposition 1.9, [15]). *The auxiliary Hamiltonian  $h(t)$  for  $P_{III}$ , as specified by (15) with parameters  $v_1, v_2$  satisfies the ordinary differential equation*

$$(16) \quad (th'')^2 = [2(h - th')] \left\{ 4(h')^2 + 16\eta_0\eta_\infty [2(h - th')] \right. \\ \left. - 16\eta_0\eta_\infty \epsilon (v_2 - v_1 - 1) \sqrt{2(h - th')} - 16\eta_0\eta_\infty (v_2 - 1/2)(v_1 + 1/2) \right\},$$

with an arbitrary sign  $\epsilon = \pm 1$ .

Proof: One can verify that the canonical variables are given by

$$\begin{aligned} 4\eta_0 p &= h' - \epsilon \frac{th''}{\sqrt{8(h - th')}} \\ 2\eta_\infty q &= \frac{h' + \epsilon \frac{th''}{\sqrt{8(h - th')}}}{v_2 - 1/2 - \epsilon \sqrt{2(h - th')}} \end{aligned}$$

with  $\epsilon = \pm 1$  provided that  $h(t)$  is not a singular solution of (16). Employing these two relations in

$$8(h - th') = (4qp - 2v_1 - 1)^2,$$

one arrives at the stated ordinary differential equation (16).  $\square$

*Remark 1.* Okamoto gives a variant of this (modulo typographical mistakes) for  $\epsilon = 1$  where it is rearranged and squared to render it polynomial.

*Remark 2.* A more restricted version of this can be found using the Ablowitz-Fokas method of Riccati transformations [2]. Thus by extending their transformation (Theorem 4.3) relating the solution  $v(z)$  of a  $P_{III}$  equation with general parameters to that of a second-order quadratic ODE with solution  $\phi(t)$ , then defining a new variable  $w(t)$  by

$$\phi^2 = 4(tw' - w) + \eta,$$

one finds  $w(t)$  satisfies a particular form of (16).

By making scale changes to dependent and independent variables (8) can be brought into correspondence to (16) and one solves for the parameters,

$$(17) \quad v_1 = -\epsilon a - 1/2, \quad v_2 = \epsilon a + 1/2$$

$$(18) \quad \eta_0 = \eta_\infty = 1, \quad r = 4it$$

$$(19) \quad \sigma_1(r) = 2h(t) - a^2 = 2tH_{III}(t).$$

Thus  $\sigma_1(r)/r$  is proportional to the Hamiltonian for the  $P_{III}$  system, with parameters as specified. Furthermore, introducing the  $\tau$ -function  $\tau_{III}(t)$  for the  $P_{III}$  system by the requirement

$$(20) \quad H_{III}(t) =: \frac{d}{dt} \log \tau_{III}(t),$$

it follows from (18), (19) and (7) that

$$(21) \quad E_2^{SS}(0; (-x, x); a) = \tau_{III}^2\left(\frac{\pi x}{2i}\right) \Big|_{v_1 = -v_2 = -\epsilon a - 1/2}.$$

So far everything is in terms of the  $P_{III}$  system, but now we will find that conversion to the  $P_{III'}$  system yields the desired  $\tau$ -function representation. The  $P_{III'}$  system  $\{s, q_{III'}, p_{III'}, H_{III'}\}$  is entirely equivalent to the  $P_{III}$  system but differs in a number of salient features, one of which is that the transcendent  $q_{III'}(s)$  has movable double poles in contrast to the single poles for  $q_{III}(t)$ .

The Hamiltonian for the  $P_{III'}$  system is given by [15]

$$(22) \quad sH_{III'}(s) = q_{III'}^2 p_{III'}^2 - (q_{III'}^2 + v_1 q_{III'} - s)p_{III'} + 1/2(v_1 + v_2)q_{III'},$$

and the associated  $\tau$ -function is given by

$$(23) \quad H_{III'} =: \frac{d}{ds} \log \tau_{III'}(s).$$

From the Hamiltonian the  $\sigma$ -function  $\sigma_{\text{III}'}$  is defined by

$$(24) \quad \sigma_{\text{III}'}(s) = -(sH_{\text{III}'})(s/4) - \frac{1}{4}v_1(v_1 - v_2) + \frac{1}{4}s.$$

The function  $\sigma_{\text{III}'}(s)$  for general  $\mathbf{v} = (v_1, v_2)$  satisfies the  $\sigma$ -form second-order second-degree ordinary differential equation

$$(25) \quad (s\sigma_{\text{III}'}'')^2 - v_1v_2(\sigma_{\text{III}'}')^2 + \sigma_{\text{III}'}'(4\sigma_{\text{III}'}' - 1)(\sigma_{\text{III}'} - s\sigma_{\text{III}'}') - \frac{1}{4^3}(v_1 - v_2)^2 = 0.$$

**Lemma 1.** *The Hamiltonian for the  $\text{P}_{\text{III}}$  system is related to that of the  $\text{P}_{\text{III}'}$  system by*

$$(26) \quad tH_{\text{III}}(t)\Big|_{\mathbf{v}} = sH_{\text{III}'}(s)\Big|_{\mathbf{v}} + sH_{\text{III}'}(s)\Big|_{T_2(\mathbf{v})},$$

where

$$(27) \quad T_2(\mathbf{v}) = (v_1 + 1, v_2 - 1),$$

and

$$(28) \quad t^2 = s.$$

*Proof.* Using the mapping relating the two systems [15]

$$(29) \quad tq_{\text{III}}(t) = q_{\text{III}'}(s)$$

$$(30) \quad \frac{1}{t}p_{\text{III}}(t) = p_{\text{III}'}(s)$$

$$(31) \quad tH_{\text{III}}(t) = 2sH_{\text{III}'}(s) - q_{\text{III}'}(s)p_{\text{III}'}(s),$$

we notice that the right-hand side of the last equality can be written

$$(32) \quad 2sH_{\text{III}'}(s) - q_{\text{III}'}p_{\text{III}'} = sH_{\text{III}'}(s) + (sH_{\text{III}'}(s) - q_{\text{III}'}p_{\text{III}'})$$

$$(33) \quad = sH_{\text{III}'}(s) + T_2(sH_{\text{III}'}(s))$$

$$(34) \quad = sH_{\text{III}'}\Big|_{(v_1, v_2)} + sH_{\text{III}'}\Big|_{(v_1+1, v_2-1)},$$

where the Schlesinger transformation  $T_2$  acts on the parameters according to (27) and upon the Hamiltonian according to  $T_2(sH_{\text{III}'}) = sH_{\text{III}'}\Big|_{\mathbf{v} \rightarrow T_2(\mathbf{v})}$  (see Table 1). Thus we have (26).  $\square$

**Proposition 1.** *The gap probability (7) is the product of two  $\tau$ -functions for the system  $\text{P}_{\text{III}'}$*

$$(35) \quad E_2^{\text{SS}}(0; (-x, x); a) = \tau_{\text{III}'}(-1/4(\pi x)^2)\Big|_{(-\epsilon a - 1/2, \epsilon a + 1/2)} \tau_{\text{III}'}(-1/4(\pi x)^2)\Big|_{(-\epsilon a + 1/2, \epsilon a - 1/2)},$$

for  $\epsilon = \pm 1$ . The relevant solutions to (25) appearing here are those with  $\mathbf{v} = (\mu, -\mu)$  and  $\mu = -\epsilon a - 1/2$  or  $\mu = -\epsilon a + 1/2$  satisfying the boundary condition

$$(36) \quad \sigma_{\text{III}'}(s) \underset{s \rightarrow 0^-}{\sim} -\frac{\mu^2}{2} + \frac{s}{4} + \tilde{C}_\mu(-s)^{1-\mu} \left( 1 - \frac{1}{2(\mu-2)}s + \frac{2\mu-3}{16(\mu-3)(\mu-2)(\mu-1)}s^2 + \dots \right) - \tilde{C}_\mu^2 \frac{(-s)^{2-2\mu}}{\mu-1} \left( 1 - \frac{2\mu-3}{2(\mu-2)^2}s + \dots \right) + \tilde{C}_\mu^3 \frac{(-s)^{3-3\mu}}{(\mu-1)^2} (1 + \dots) + \dots,$$

where

$$(37) \quad \tilde{C}_\mu = \frac{1}{4^{-\mu+1}\Gamma(-\mu+2)\Gamma(-\mu+1)}.$$

There is another known product form of (7) [4],[3] which states

$$(38) \quad E_2^{\text{SS}}(0; (-x, x); a) = E_2^{\text{HE}}(0; (0, \pi^2 x^2); a - \tfrac{1}{2}) E_2^{\text{HE}}(0; (0, \pi^2 x^2); a + \tfrac{1}{2}),$$

where the hard edge gap probability has the evaluation

$$(39) \quad E_2^{\text{HE}}(0; (0, X); a) = e^{-X/4} \tau_{\text{III}'}(\tfrac{1}{4}X) \Big|_{\mathbf{v}=(a,a)} = \exp \left( - \int_0^X \frac{ds}{s} \sigma_{\text{III}'}(s) \right) \Big|_{\mathbf{v}=(a,a)},$$

again in terms of  $\tau$ -functions for the  $\text{P}_{\text{III}'}$  system. The particular solution of (25) arising in the hard edge case satisfies the boundary condition

$$(40) \quad \sigma_{\text{III}'}(s) \underset{s \rightarrow 0^+}{\sim} C_a s^{a+1} \left( 1 - \frac{1}{2(a+2)} s + \frac{2a+3}{16(a+3)(a+2)(a+1)} s^2 + \dots \right) \\ + C_a^2 \frac{s^{2a+2}}{a+1} \left( 1 - \frac{2a+3}{2(a+2)^2} s + \dots \right) + C_a^3 \frac{s^{3a+3}}{(a+1)^2} (1 + \dots) + \dots,$$

where

$$(41) \quad C_a = \frac{1}{2^{2a+2}\Gamma(a+2)\Gamma(a+1)}.$$

Combining these two product forms we have the general identity

$$(42) \quad e^{-2X} \tau_{\text{III}'}(X) \Big|_{(a-\frac{1}{2}, a-\frac{1}{2})} \tau_{\text{III}'}(X) \Big|_{(a+\frac{1}{2}, a+\frac{1}{2})} \\ = \tau_{\text{III}'}(-X) \Big|_{(-\epsilon a - \frac{1}{2}, \epsilon a + \frac{1}{2})} \tau_{\text{III}'}(-X) \Big|_{(-\epsilon a + \frac{1}{2}, \epsilon a - \frac{1}{2})},$$

or an additive relation in terms of the Hamiltonian functions,

$$(43) \quad -2s + sH_{\text{III}'}(s) \Big|_{(a-\frac{1}{2}, a-\frac{1}{2})} + sH_{\text{III}'}(s) \Big|_{(a+\frac{1}{2}, a+\frac{1}{2})} \\ = sH_{\text{III}'}(-s) \Big|_{(-\epsilon a - \frac{1}{2}, \epsilon a + \frac{1}{2})} + sH_{\text{III}'}(-s) \Big|_{(-\epsilon a + \frac{1}{2}, \epsilon a - \frac{1}{2})}.$$

A direct derivation of the latter relation can be deduced from the actions of the reflection operators  $s_2$  and  $s_-s_2$  which have the actions  $s_2 : v_2 \mapsto -v_2, s \mapsto -s$  and  $s_-s_2 : v_1 \mapsto -v_1, s \mapsto -s$ . Thus from Table 1 we see that

$$(44) \quad sH_{\text{III}'}(-s) \Big|_{(-v_1, v_2)} = s_-s_2 sH_{\text{III}'}(s) \Big|_{(v_1, v_2)} = sH_{\text{III}'}(s) \Big|_{(v_1, v_2)} - s$$

$$(45) \quad sH_{\text{III}'}(-s) \Big|_{(v_1, -v_2)} = s_2 sH_{\text{III}'}(s) \Big|_{(v_1, v_2)} = sH_{\text{III}'}(s) \Big|_{(v_1, v_2)} - s,$$

and (43) follows by adding two instances of either of the above two relations.

There is an interesting special case when the  $\tau$ -functions are classical solutions and this was found to occur in the studies [15],[7] for  $a \in \mathbb{Z}_{\geq 0} + \frac{1}{2}$ . For the  $\tau$ -functions with parameters on the diagonal ( $a = n - \frac{1}{2}, n \in \mathbb{Z}_{>0}$ )

$$(46) \quad \tau_{\text{III}'}(X) \Big|_{(a+\frac{1}{2}, a+\frac{1}{2})} = \det \left[ I_{j-k}(2\sqrt{X}) \right]_{j,k=0,\dots,n-1},$$

whereas on the cross-diagonal

$$(47) \quad \tau_{III'}(X) \Big|_{(-a-1/2, a+1/2)} = e^X \det \left[ J_{j-k}(2\sqrt{X}) \right]_{j,k=0,\dots,n-1},$$

where  $J_\nu(z)$ ,  $I_\nu(z)$  are the Bessel function and modified Bessel function respectively. Naturally the identity (42) is satisfied for these two sets of classical solution.

In conclusion our results raise the natural question whether a similar evaluation in terms of  $\tau$ -function products as in (35) should exist for the gap probabilities defined on symmetric double intervals with the finite  $N$  ensembles of Gaussian, symmetric Jacobi and Cauchy weights. These are known to be solved in terms of  $P_V$  and  $P_{VI}$  transcendents and there are analogues of the higher degree ODE (8). Moreover the analogue of the product formula (38) is known for each of these cases [4], [3]. The next question is whether there is a corresponding Hamiltonian theory underlying these ODEs and how it relates to the known Hamiltonian theory. Isolated examples of higher degree second-order ODEs (quadratic and quartic in the second derivative) are known in the literature through many differing approaches [17], [14],[10],[16], and one may suspect that these too appear in a random matrix context.

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	$v_1$	$v_2$	$p$	$q$	$s$	$sH$
$s_0$	$-1 - v_2$	$-1 - v_1$	$\frac{q}{t} [q(p-1) - \frac{1}{2}(v_1 - v_2)] + 1$	$-\frac{t}{q}$	$s$	$sH - q(p-1) + \frac{1}{2}(v_1 - v_2)(1 + \frac{1}{2}(v_1 + v_2))$
$s_1$	$v_2$	$v_1$	$p$	$q + \frac{v_2 - v_1}{2(p-1)}$	$s$	$sH - \frac{1}{4}(v_2^2 - v_1^2)$
$s_2$	$v_1$	$-v_2$	$1 - p$	$-q$	$-s$	$sH - s$
$s_-$	$-v_1$	$-v_2$	$p$	$q - \frac{v_1 p - \frac{1}{2}(v_1 + v_2)}{p(p-1)}$	$s$	$sH$
$T_2$	$v_1 + 1$	$v_2 - 1$	$\frac{q}{t} [\frac{1}{2}(v_1 + v_2) - qp]$	$\frac{t}{q} - \frac{\frac{1}{2}(2 + v_1 - v_2)t}{q[qp - \frac{1}{2}(v_1 + v_2)] + t}$	$s$	$sH - qp$

TABLE 1. Generators and selected elements of the group of Bäcklund transformations for the PIV system, the extended affine Weyl group for the root system  $B_2^{(1)}$ .